Free-field Representations and Geometry of some Gepner models.

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Abstract

The geometry of k^K Gepner model, where $k+2=2K$ is investigated by free-field representation known as "$bc\beta\gamma$"-system. Using this representation it is shown directly that internal sector of the model is given by Landau-Ginzburg $\mathbb{C}^K/\mathbb{Z}_{2K}$-orbifold. Then we consider the deformation of the orbifold by marginal anti-chiral-chiral operator. Analyzing the holomorphic sector of the deformed space of states we show that it has chiral de Rham complex structure of some toric manifold, where toric dates are given by certain fermionic screening currents. It allows to relate the Gepner model deformed by the marginal operator to the σ-model on CY manifold realized as double cover of $\mathbb{P}^{K-1}$ with ramification along certain submanifold.

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1. Introduction

Geometric aspects underlying purely algebraic, Conformal Field Theory (CFT) construction of Gepner [1] of the superstring vacua is an important and interesting area of study. It has two decades history of research with a number of bright results. In consequence of this the relationship between the σ-models on Calabi-Yau (CY) manifolds and Gepner models has been clarified essentially. For the review and the references on the original papers see [2].

However the question how to relate directly the σ-model geometry to the algebraic dates of Gepner's construction and when it is possible is still open.

In the important work of Borisov [3] the vertex operator algebra endowed with $N=2$ Virasoro superalgebra action has been constructed for each pair of dual reflexive polytopes defining toric CY manifold. Thus, Borisov constructed directly holomorphic sector of the CFT from toric dates of CY manifold. His approach is based essentially on the important work of Malikov, Schechtman and Vaintrob [4] where a certain sheaf of vertex algebras which is called chiral de Rham complex has been introduced. Roughly speaking the construction of [4] is a kind of free-field representation known as "$bc\beta\gamma$"-system which is in case of Gepner model closely related with the Feigin and Semikhatov free-field representation [7] of $N=2$ supersymmetric minimal models. This circumstance is probably the key for understanding string geometry of Gepner models and their relationship to the σ-models on toric CY manifolds.

The significant step in this direction has been made in the paper [5] where the vertex algebra of certain Landau-Ginzburg (LG) orbifold has been related to chiral de Rham complex of toric CY manifold by a spectral sequence. The CY manifold has been realized as an algebraic surface

degree K in the projective space $\mathbb{P}^{K-1}$ and one of the key points of [5] is that the free-field representation of the corresponding LG orbifold is given by K copies of $N = 2$ minimal model free-field representation of [7].

The Gepner model can be characterized by K-dimensional vector

$$\boldsymbol{\mu} = (\mu_1, ..., \mu_K) \tag{1}$$

where

$$\mu_i = 2, 3, ..., \;\; i = 1, ..., K \tag{2}$$

define the central charges of the individual $N = 2$ minimal models

$$c_i = 3(1 - \frac{2}{\mu_i}) \tag{3}$$

In what follows the μ_i will be specified by

$$\boldsymbol{\mu} = (\mu, \mu, ..., \mu) \tag{4}$$

so the total central charge of the model is

$$c = \sum_{i=1}^{K} c_i = 3K(1 - \frac{2}{\mu}) \tag{5}$$

There are two cases when the central charge is integer and multiple of 3

$$\mu = K, \;\; 2K \tag{6}$$

The geometry underlying the first case has been investigated in [5].

In the second case the geometry is more interesting. The total central charge is

$$c = 3(K - 1) \tag{7}$$

and hence the complex dimension of the compact manifold is $K - 1$. I am going to show in this note that the internal geometry of Gepner model corresponds in this case to the σ-model on the CY manifold which double covers the $\mathbb{P}^{K-1}$ with ramification along certain submanifold. It means in particular that center of mass of the string is allowed to move only along the base $\mathbb{P}^{K-1}$ but with some twisted sectors added along the fiber of the double cover.

One can generalize the second case and consider the models where

$$\mu = 3K, 4K, ... \tag{8}$$

Though the total central charge is no longer integer and these models can not be used as the models of superstring compactification, the orbifold projection consistent with modular invariance still exits [6] which makes them to be interesting $N = 2$ supersymmetic models of CFT from the geometric point of view. The geometry of these models has been investigated partly in [8].

In Section 2 I represent a collection of known facts on the $N = 2$ minimal models, fix the notations and briefly remind the Gepner's construction of the partition function in the internal

sector of the Gepner model. In Section 3 the free-field representation of [7] is used to relate the model with LG $\mathbb{C}^K/\mathbb{Z}_{2K}$-orbifold.

In Sect.4 the resolution of the orbifold singularity in chiral sector is considered. It is given by adding some new fermionic screening charge coming from the twisted sector of Gepner model. We show that this additional screening charge together with the old charges define the toric dates of $O(K)$-bundle total space over the $\mathbb{P}^{K-1}$ as well as the potential on this space. The chiral sector space of states of the model has the chiral de Rham complex structure on the $O(K)$-bundle total space restricted to zeroes of the gradient of the potential. Then we consider the rest of the orbifold group action on the space of states and relate the model with σ-model on a CY manifold which double covers the projective space $\mathbb{P}^{K-1}$.

2. The internal sector partition function of the Gepner models.

In this section we remind the construction of the partition function of the Gepner model in the internal sector. To be more specific the Ramound-Ramound (RR) partition function of the internal sector will be important for the geometry investigation. But as a preliminary we represent a collection of known facts on the $N=2$ minimal models and fix the notations.

2.1. The products of $N=2$ minimal models.

The tensor product of K $N=2$ unitary minimal models can be characterized by K-dimensional vector $\boldsymbol{\mu}=(\mu_1,...,\mu_K)$, where $\mu_i\geq 2$ being integer defines the central charge of the individual model by $c_i=3(1-\frac{2}{\mu_i})$. For each individual minimal model we denote by $M_{h,t}$ the irreducible unitary $N=2$ Virasoro superalgebra representation in NS sector and denote by $\chi_{h,t}(q,u)$ the character of the representation

$$\chi_{h,t}(q,u)=Tr_{h,t}(q^{L[0]-\frac{c}{24}}u^{J[0]}) \tag{9}$$

where $h=0,...,\mu-2$ and $t=0,...,h$. There are the following important automorphisms of the irreducible modules and characters [7], [9].

$$M_{h,t}\equiv M_{\mu-h-2,t-h-1},\ \chi_{h,t}(q,u)=\chi_{\mu-h-2,t-h-1}(q,u), \tag{10}$$

$$M_{h,t}\equiv M_{h,t+\mu},\ \chi_{h,t+\mu}(q,u)=\chi_{h,t}(q,u), \tag{11}$$

where μ is odd and

$$\begin{aligned} M_{h,t}\equiv M_{h,t+\mu},\ \chi_{h,t+\mu}(q,u)=\chi_{h,t}(q,u),\ h\neq\frac{\mu}{2}-1,\\ M_{h,t}\equiv M_{h,t+\frac{\mu}{2}},\ \chi_{h,t+\frac{\mu}{2}}(q,u)=\chi_{h,t}(q,u),\ h=\frac{\mu}{2}-1, \end{aligned} \tag{12}$$

where μ is even. In what follows we extend the set of admissible t:

$$t=0,...,\mu-1 \tag{13}$$

using the automorphisms above.

The parameter $t\in\mathbb{Z}$ labels the spectral flow automorphisms [10] of $N=2$ Virasoro superalgebra in NS sector

$$\begin{aligned} G^{\pm}[r]\rightarrow G^{\pm}_t[r]\equiv U^tG^{\pm}[r]U^{-t}\equiv G^{\pm}[r\pm t],\\ L[n]\rightarrow L_t[n]\equiv U^tL[n]U^{-t}\equiv L[n]+tJ[n]+t^2\frac{c}{6}\delta_{n,0},\\ J[n]\rightarrow J_t[n]\equiv U^tJ[n]U^{-t}\equiv J[n]+t\frac{c}{3}\delta_{n,0}, \end{aligned} \tag{14}$$

where U^t denotes the spectral flow operator generating twisted sectors. Here r is half-integer for the modes of the spin-3/2 fermionic currents $G^{\pm}(z)$ while n is integer for the modes of stress-energy tensor $T(z)$ and $U(1)$-current $J(z)$ of the $N=2$ Virasoro superalgebra. So allowing t to be half-integer we recover the irreducible representations and characters in the R sector.

The $N=2$ Virasoro superalgebra generators in the product of minimal models are given by the sums of generators of each minimal model

$$
\begin{aligned}
G^{\pm}[r] &= \sum_i G_i^{\pm}[r], \\
J[n] = \sum_i J_i[n],\ T[n] &= \sum_i T_i[n], \\
c &= \sum_i 3(1-\frac{2}{\mu_i})
\end{aligned}
\tag{15}
$$

This algebra is obviously acting in the tensor product $M_{\mathbf{h},\mathbf{t}} = \otimes_{i=1}^{K} M_{h_i,t_i}$ of the irreducible $N=2$ Virasoro superalgebra representations of each individual model. We use the similar notation for the corresponding product of characters

$$
\chi_{\mathbf{h},\mathbf{t}}(q,u) = \prod_{i=1}^{K} \chi_{h_i,t_i}(q,u) \tag{16}
$$

2.2. The partition function of the internal sector.

In what follows the characters with fermionic number operator insertion will be important

$$
\tilde{\chi}_{h_i,t_i}(q,u) = Tr_{h_i,t_i}((-1)^F q^{L_i[0]-\frac{c_i}{24}} u^{J_i[0]}). \tag{17}
$$

The internal sector partition function of the Gepner model in RR-sector is given by

$$
Z(q,\bar{q},u,\bar{u}) = \frac{1}{2K2^K} \sum_{n,m}^{2K-1} \prod_{i=1}^{K} \sum_{h_i,t_i} \tilde{\chi}_{h_i,t_i+n+\frac{1}{2}}(\tau,\upsilon+m) \tilde{\chi}^*_{h_i,+t_i+\frac{1}{2}}(\tau,\upsilon) \tag{18}
$$

where $q=\exp[\imath 2\pi\tau]$, $u=\exp[\imath 2\pi\upsilon]$ and $*$ denotes the complex conjugation. The summation over n is due to the spectral flow twisted sector generated by the product of spectral flow operators $\prod_{i=1}^{K} U_i^n$. The summation over m corresponds to the projection on the $\mathbb{Z}_{2K}$-invariant states with respect to the operator $\exp[\imath 2\pi J[0]]$. Thus it is $\mathbb{Z}_{2K}$-orbifold partition function in RR-sector with periodic spin structure along the both cycles of the torus.

3. Free-field representations and LG orbifold geometry of Gepner models.

In this section we relate the Gepner models to the LG orbifolds $\mathbb{C}^K/\mathbb{Z}_{2K}$ using essentially the free-field construction of irreducible representations of $N=2$ minimal models found by Feigin and Semikhatov in [7].

3.1. Free-field realization of $N=2$ minimal model.

Let $X(z), X^*(z)$ be the free bosonic fields and $\psi(z), \psi^*(z)$ be the free fermionic fields (in the left-moving sector) so that its OPE's are given by

$$
\begin{aligned}
X^*(z_1)X(z_2) &= \ln(z_{12}) + reg., \\
\psi^*(z_1)\psi(z_2) &= z_{12}^{-1} + reg,
\end{aligned}
\tag{19}
$$

where $z_{12} = z_1 - z_2$. For an arbitrary number μ the currents of $N = 2$ super-Virasoro algebra are given by

$$G^{+}(z) = \psi^{*}(z)\partial X(z) - \frac{1}{\mu}\partial\psi^{*}(z),\ G^{-}(z) = \psi(z)\partial X^{*}(z) - \partial\psi(z),$$
$$J(z) = \psi^{*}(z)\psi(z) + \frac{1}{\mu}\partial X^{*}(z) - \partial X(z),$$
$$T(z) = \partial X(z)\partial X^{*}(z) + \frac{1}{2}(\partial\psi^{*}(z)\psi(z) - \psi^{*}(z)\partial\psi(z)) - \frac{1}{2}(\partial^{2}X(z) + \frac{1}{\mu}\partial^{2}X^{*}(z)), \tag{20}$$

and the central charge is

$$c = 3(1 - \frac{2}{\mu}). \tag{21}$$

As usual, the fermions are expanded into the half-integer modes in NS sector and they are expanded into integer modes in R sector

$$\psi(z) = \sum_{r}\psi[r]z^{-\frac{1}{2}-r},\ \psi^{*}(z) = \sum_{r}\psi^{*}[r]z^{-\frac{1}{2}-r},\ G^{\pm}(z) = \sum_{r}G^{\pm}[r]z^{-\frac{3}{2}-r}, \tag{22}$$

The bosons are expanded in both sectors into the integer modes:

$$\partial X(z) = \sum_{n\in Z}X[n]z^{-1-n},\ \partial X^{*}(z) = \sum_{n\in Z}X^{*}[n]z^{-1-n},$$
$$J(z) = \sum_{n\in Z}J[n]z^{-1-n},\ T(z) = \sum_{n\in Z}L[n]z^{-2-n}. \tag{23}$$

In NS sector $N = 2$ Virasoro superalgebra is acting naturally in Fock module $F_{p,p^{*}}$ generated by the fermionic operators $\psi^{*}[r]$, $\psi[r]$, $r < \frac{1}{2}$, and bosonic operators $X^{*}[n]$, $X[n]$, $n < 0$ from the vacuum state $|p, p^{*}>$ such that

$$\psi[r]|p,p^{*}>= \psi^{*}[r]|p,p^{*}>= 0, r \geq \frac{1}{2},$$
$$X[n]|p,p^{*}>= X^{*}[n]|p,p^{*}>= 0, n \geq 1,$$
$$X[0]|p,p^{*}>= p|p,p^{*}>,\ X^{*}[0]|p,p^{*}>= p^{*}|p,p^{*}>. \tag{24}$$

It is a primary state with respect to the $N = 2$ Virasoro algebra

$$G^{\pm}[r]|p,p^{*}>= 0, r > 0,$$
$$J[n]|p,p^{*}>= L[n]|p,p^{*}>= 0, n > 0,$$
$$J[0]|p,p^{*}>= \frac{j}{\mu}|p,p^{*}>= 0,$$
$$L[0]|p,p^{*}>= \frac{h(h+2) - j^{2}}{4\mu}|p,p^{*}>= 0, \tag{25}$$

where $j = p^{*} - \mu p$, $h = p^{*} + \mu p$.

When $\mu - 2$ is integer and non negative the Fock module is highly reducible representation of $N = 2$ Virasoro algebra.

The irreducible module $M_{h,j}$ is given by cohomology of some complex building up from Fock modules. This complex has been constructed in [7]. Let us consider first free-field construction for the chiral module $M_{h,0}$. In this case the complex (which is known due to Feigin and Semikhatov as butterfly resolution) can be represented by the following diagram

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & & & & & \\
 & & \uparrow & & \uparrow & & & & & & \\
\dots & \leftarrow & F_{1,h+\mu} & \leftarrow & F_{0,h+\mu} & & & & & & \\
 & & \uparrow & & \uparrow & & & & & & \\
\dots & \leftarrow & F_{1,h} & \leftarrow & F_{0,h} & & & & & & \\
 & & & & & \nwarrow & & & & & \\
 & & & & & & F_{-1,h-\mu} & \leftarrow & F_{-2,h-\mu} & \leftarrow & \dots \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & F_{-1,h-2\mu} & \leftarrow & F_{-2,h-2\mu} & \leftarrow & \dots \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & \vdots & & \vdots & &
\end{array}
\tag{26}
$$

The horizontal arrows in this diagram are given by the action of

$$Q^{+} = \oint dz S^{+}(z),\ \ S^{+}(z) = \psi^{*} \exp(X^{*})(z), \tag{27}$$

The vertical arrows are given by the action of

$$Q^{-} = \oint dz S^{-}(z),\ \ S^{-}(z) = \psi \exp(\mu X)(z), \tag{28}$$

The diagonal arrow at the middle of butterfly resolution is given by the action of $Q^{+}Q^{-}$. It is a complex due to the following properties screening charges $Q^{\pm}$

$$(Q^{+})^{2} = (Q^{-})^{2} = \{Q^{+}, Q^{-}\} = 0. \tag{29}$$

The main statement of [7] is that the complex (26) is exact except at the $F_{0,h}$ module, where the cohomology is given by the chiral module $M_{h,0}$.

To get the resolution for the irreducible module $M_{h,t}$ one can use the observation [7] that all irreducible modules can be obtained from the chiral module $M_{h,0},\ h = 0, ..., \mu - 2$ by the spectral flow action $U^{-t}, t = 1, ..., \mu - 1$. The spectral flow action on the free fields can be easily described if we bosonize fermions ψ^{*}, ψ

$$\psi(z) = \exp(-\phi(z)),\ \ \psi^{*}(z) = \exp(\phi(z)). \tag{30}$$

and introduce spectral flow vertex operator

$$U^{t}(z) = \exp(-t(\phi + \frac{1}{\mu}X^{*} - X)(z)). \tag{31}$$

Using the resolution (26) and the spectral flow we obtain the following expression for the character [9]

$$\chi_{h,-t}(u,q)=q^{\frac{h}{2\mu}+\frac{c}{6}t^2+\frac{th}{\mu}-\frac{c}{24}}q^{\frac{1-\mu}{8}}u^{\frac{h}{\mu}+\frac{ct}{3}}(\frac{\eta(q^{\mu})}{\eta(q)})^3$$
$$\prod_{n=0}\frac{(1+uq^{\frac{1}{2}+t+n})}{(1+u^{-1}q^{-\frac{1}{2}-t+n\mu})}\frac{(1+u^{-1}q^{\frac{1}{2}-t+n})}{(1+uq^{\frac{1}{2}+t+(n+1)\mu})}\frac{(1-q^{n+1})}{(1-q^{(n+1)\mu})}$$
$$\prod_{n=0}\frac{(1-q^{-1-h+n\mu})}{(1+uq^{-\frac{1}{2}-h+t+n\mu})}\frac{(1-q^{1+h+(n+1)\mu})}{(1+u^{-1}q^{\frac{1}{2}+h-t+(n+1)\mu})} \tag{32}$$

where

$$\eta(q)=q^{\frac{1}{24}}\prod_{n=1}(1-q^n) \tag{33}$$

The resolutions and irreducible modules in R sector are generated from the resolutions and modules in NS sector by the spectral flow operator $U^{\frac{1}{2}}$.

3.2.Free-field realization of the product of minimal models.

It is clear how to generalize the free-field representation to the case of tensor product of K $N=2$ minimal models. One has to introduce (in the left-moving sector) the free bosonic fields $X_i(z),X_i^*(z)$ and free fermionic fields $\psi_i(z),\psi_i^*(z)$, $i=1,...,K$ so that its singular OPE's are given by (19). The $N=2$ superalgebra Virasoro currents for each of the models are given by (20). To describe the products of irreducible representations $M_{\mathbf{h},\mathbf{t}}$ we introduce the fermionic screening currents and their charges

$$S_i^+(z)=\psi_i^*\exp(X_i^*)(z),\ S_i^-(z)=\psi_i\exp(\mu_iX_i)(z),\ Q_i^{\pm}=\oint dzS_i^{\pm}(z). \tag{34}$$

Then the module $M_{\mathbf{h},0}$ is given by the cohomology of the product of butterfly resolutions (26). The resolution of the module $M_{\mathbf{h},\mathbf{t}}$ is generated by the spectral flow operator $U^{\mathbf{t}}=\prod_i U_i^{t_i}$, $t_i=1,...,\mu_i-1$, where $U_i^{t_i}$ is the spectral flow operator from the i-th minimal model (31). Allowing t_i to be half-integer we generate the corresponding objects in R sector. In what follows we consider the case $\mu_1=...=\mu_K=2K$.

3.3. LG orbifold geometry of Gepner models.

The holomorphic factor of the space of states of the model (18) in R-sector is given also by cohomology of the complex. It is an orbifold of the complex which is the sum of butterfly resolutions for the modules $M_{\mathbf{h},\mathbf{t}}$. The cohomology of this complex can be calculated by two steps.

At first step we take the cohomology wrt the operator

$$Q^+=\sum_{i=1}^{K}Q_i^+ \tag{35}$$

It is generated by $bc\beta\gamma$ system of fields

$$a_i(z)=\exp[X_i](z),\ \alpha_i(z)=\psi_i\exp[X_i](z),$$
$$a_i^*(z)=(\partial X_i^*-\psi_i\psi_i^*)\exp[-X_i](z),\ \alpha_i^*(z)=\psi_i^*\exp[-X_i](z) \tag{36}$$

The singular operator product expansions of these fields are

$$a^*_i(z_1)a_j(z_2) = z^{-1}_{12}\delta_{ij} + ...,$$
$$\alpha^*_i(z_1)\alpha_j(z_2) = z^{-1}_{12}\delta_{ij} + ... \quad (37)$$

In terms of the fields (36) the N=2 Virasoro superalgebra currents (15) are given by

$$G^- = \sum_i \alpha_i a^*_i,\ G^+ = \sum_i (1-\frac{1}{2K})\alpha^*_i \partial a_i - \frac{1}{2K} a_i \partial \alpha^*_i,$$
$$J = \sum_i (1-\frac{1}{2K})\alpha^*_i \alpha_i + \frac{1}{2K} a_i a^*_i,$$
$$T = \sum_i \frac{1}{2}((1+\frac{1}{2K})\partial\alpha^*_i\alpha_i - (1-\frac{1}{2K})\alpha^*_i\partial\alpha_i) + (1-\frac{1}{4K})\partial a_i a^*_i - \frac{1}{4K} a_i \partial a^*_i \quad (38)$$

Notice that zero mode $G^-[0]$ is acting in the space of states generated by $bc\beta\gamma$ system of fields similar to the de Rham differential action in the de Rham complex of $\mathbb{C}^K$. Due to this observation and taking into account (37) one can make the following geometric interpretation of the fields (36). The fields $a_i(z)$ correspond to the coordinates a_i on the complex space $\mathbb{C}^K$, the fields $a^*_i(z)$ correspond to the operators $\frac{\partial}{\partial a_i}$. The fields $\alpha_i(z)$ correspond to the differentials da_i, while $\alpha^*_i(z)$ correspond to the conjugated to da_i.

The next important property is the behaviour of the $bc\beta\gamma$ system under the local change of coordinates on $\mathbb{C}^K$ [4]. For each new set of coordinates

$$b_i = g_i(a_1, ..., a_K),\ a_i = f_i(b_1, ..., b_K) \quad (39)$$

the isomorphic $bc\beta\gamma$ system of fields is given by

$$b_i(z) = g_i(a_1(z), ..., a_K(z)),$$
$$\beta_i(z) = \frac{\partial g_i}{\partial a_j}(a_1(z), ..., a_K(z))\alpha_j(z),$$
$$\beta^*_i(z) = \frac{\partial f_j}{\partial b_i}(a_1(z), ..., a_K(z))\alpha^*_j(z),$$
$$b^*_i(z) = \frac{\partial f_j}{\partial b_i}(a_1(z), ..., a_K(z))a^*_j(z) +$$
$$\frac{\partial^2 f_k}{\partial b_i \partial b_j}\frac{\partial g_j}{\partial a_n}(a_1(z), ..., a_K(z))\alpha^*_k(z)\alpha_n(z) \quad (40)$$

where the normal ordering of the fields is implied. It endows the $bc\beta\gamma$ system (36) with the structure of sheaf known as chiral de Rham complex due to [4].

All these properties provide the geometric meaning to the algebraic construction of Gepner model. Indeed, it was shown by Borisov in general toric setup [3] that the screening charges Q^+_i determine the toric dates of some toric manifold and the cohomology of the differentail (35) gives the sections of chiral de Rham complex on this manifold. In our case this manifold is $\mathbb{C}^K$ and chiral de Rham complex on this space is generated by $bc\beta\gamma$ system (36).

The charges of the fields (36) are given by

$$J(z_1)a_i(z_2) = z^{-1}_{12}\frac{1}{2K}a_i(z_2) + r.,\ J(z_1)a^*_i(z_2) = -z^{-1}_{12}\frac{1}{2K}a^*_i(z_2) + r.,$$
$$J(z_1)\alpha_i(z_2) = -z^{-1}_{12}(1-\frac{1}{2K})\alpha_i(z_2) + r.,\ J(z_1)\alpha^*_i(z_2) = z^{-1}_{12}(1-\frac{1}{2K})\alpha^*_i(z_2) + r. \quad (41)$$

Hence, making the projection on $\mathbb{Z}_{2K}$-invariant states and adding twisted sectors generated by $\prod_{i=1}^{K}(U_i)^n$ we obtain toric construction of the chiral de Rham complex of the orbifold $\mathbb{C}^K/\mathbb{Z}_{2K}$. The chiral de Rham complex on the orbifold has recently been introduced in [11].

The second step in the cohomology calculation is given by the cohomology with respect to the differential $Q^- = \sum_{i=1}^{K} Q_i^-$. This operator survives the orbifold projection and its expression in terms of fields (36) is

$$Q^- = \oint dz \sum_{i=1}^{K} \alpha_i (a_i)^{2K-1} \tag{42}$$

Therefore the second step of cohomology calculation gives the restriction of the chiral de Rham complex to the points $dW = 0$ of the potential

$$W = \sum_{i=1}^{K} (a_i)^{2K} \tag{43}$$

Thus the total space of states is the space of states of LG orbifold $\mathbb{C}^K/\mathbb{Z}_{2K}$ whose partition function in RR-sector is given by (18).

4. LG/sigma-model correspondence conjecture.

In this section we relate the LG orbifold $\mathbb{C}^K/\mathbb{Z}_{2K}$ to the σ-model on CY manifold which double cover the space $\mathbb{P}^{K-1}$. The relation appears when we deform LG-orbifold by marginal operator making the orbifold singularity resolution. According to the construction [3], [5], the orbifold singularity resolution in holomorphic sector is given by supplementary screening charges.

4.1. $K = 2$ example.

Let us consider first this procedure in the simplest example $K = 2$. In this case we add to the charges $Q_{1,2}^+$ the screening charge

$$D_{orb} = \oint dz \frac{1}{2}(\psi_1^* + \psi_2^*) \exp(\frac{1}{2}(X_1^* + X_2^*))(z) \tag{44}$$

It is easy to check that this operator commutes with the total $N = 2$ Virasoro superalgebra currents (15) and commutes also with the operators Q_i^-. The corresponding fermionic screening current is the holomorphic (chiral) factor of the anti-chiral-chiral marginal field [13], [2], coming from the twisted sector. The fermionic operators

$$D_n^+ = \oint dz (\frac{2-n}{4}\psi_1^* + \frac{2+n}{4}\psi_2^*) \exp(\frac{2-n}{4}X_1^* + \frac{2+n}{4}X_2^*)(z), \ n = -1, 1 \tag{45}$$

also commute with $N = 2$ Virasoro algebra and Q_i^- but they do not appear as marginal operators of the model because they should come from twisted sectors which are not exist in the model (see (18)).

Following the construction of Borisov we associate to the set of screening charges Q_1^+, Q_2^+, D_{orb} the fan [12] consisting of two 2-dimensional cones σ_1 and σ_2, generated in the lattice $(\frac{1}{2}\mathbb{Z})^2$ by the vectors $(e_1, \frac{1}{2}(e_1 + e_2))$ and vectors $(e_2, \frac{1}{2}(e_1 + e_2))$ correspondingly. To each of the cones σ_i the $bc\beta\gamma$ system of fields is related by the cohomology of the differential $Q_i^+ + D_{orb}$, $i = 1, 2$. This is the first step of cohomology calculation.

One can show that these two systems generate the space of sections of the chiral de Rham complex on the open sets of the standard covering of the total space of $O(2)$-bundle over $\mathbb{P}^1$.

Indeed, one can split the first step of cohomology calculation into 2 substeps. At the first substep we take $Q_1^+ + D_{orb}$-cohomology. It is given by the following $bc\beta\gamma$ fields

$$\begin{aligned}
b_0(z) = \exp[2X_2](z),\ \beta_0(z) = 2\psi_2 \exp[2X_2](z),\\
b_0^*(z) = (\frac{1}{2}(\partial X_1^* + \partial X_2^*) - \psi_2(\psi_1^* + \psi_2^*)) \exp[-2X_2](z),\ \beta_0^*(z) = \frac{1}{2}(\psi_1^* + \psi_2^*) \exp[-2X_2](z),\\
b_1(z) = \exp[X_1 - X_2](z),\ \beta_1(z) = (\psi_1 - \psi_2) \exp[X_1 - X_2](z),\\
b_1^*(z) = (\partial X_1^* - (\psi_1 - \psi_2)\psi_1^*) \exp[X_2 - X_1](z),\ \beta_1^*(z) = \psi_1^* \exp[X_2 - X_1](z)
\end{aligned} \tag{46}$$

At the second substep we calculate Q_2^+-cohomology.

On the equal footing one can take $Q_2^+ + D_{orb}$-cohomology as the first substep and apply Q_1^+ at the second substep. Going by this way we obtain another $bc\beta\gamma$ fields:

$$\begin{aligned}
\tilde{b}_0(z) = \exp[2X_1](z),\ \tilde{\beta}_0(z) = 2\psi_1 \exp[2X_1](z),\\
\tilde{b}_0^*(z) = (\frac{1}{2}(\partial X_1^* + \partial X_2^*) - \psi_1(\psi_1^* + \psi_2^*)) \exp[-2X_1](z),\\
\tilde{\beta}_0^*(z) = \frac{1}{2}(\psi_1^* + \psi_2^*) \exp[-2X_1](z),\\
\tilde{b}_1(z) = \exp[X_2 - X_1](z),\ \tilde{\beta}_1(z) = (\psi_2 - \psi_1) \exp[X_2 - X_1](z),\\
\tilde{b}_1^*(z) = (\partial X_2^* - (\psi_2 - \psi_1)\psi_2^*) \exp[X_1 - X_2](z),\\
\tilde{\beta}_1^*(z) = \psi_2^* \exp[X_1 - X_2](z)
\end{aligned} \tag{47}$$

In view of the important property (40) these two $bc\beta\gamma$ systems are related to each other like the coordinates of the standard covering of the total space of $O(2)$-bundle over $\mathbb{P}^1$ do

$$b_0 = \tilde{b}_0(\tilde{b}_1)^2,\ b_1 = \tilde{b}_1^{-1}, ... \tag{48}$$

Therefore

$$\begin{aligned}
b_0(z) \leftrightarrow\ coordinate\ along\ the\ fiber\ b_0,\\
b_1(z) \leftrightarrow\ coordinate\ along\ the\ base\ b_1
\end{aligned} \tag{49}$$

in the first open set of the standard covering. The tilda-fields service the second open set. Thus, the fields (46) and (47) generate the sections of the chiral de Rham complex over the open sets of the covering given by the fan $\sigma_1 \cup \sigma_2$. Doing the second substep we calculate the cohomology of the Chech complex of the standard covering. It glues the sections of chiral de Rham complex over the open sets into the chiral de Rham complex over the total space of the bundle. It is the end of the first step of the cohomology calculation.

The differential Q^- of the second step cohomology calculation commutes with D_{orb} and survives $\mathbb{Z}_4$-projection. It defines the function (potential) W on the total space of $O(2)$-bundle and Q^--cohomology calculation restricts the chiral de Rham complex to the $dW = 0$ point set of the function. In terms of the fields (46) the potential takes the form

$$W = b_0^2(1 + b_1^4) \tag{50}$$

The $dW = 0$ points (Q^--cohomology) are given by the equations

$$\begin{aligned}
b_0 = 0,\ when\ b_1^4 \neq -1,\\
(b_0)^2 = 0,\ when\ b_1^4 = -1,
\end{aligned} \tag{51}$$

The set of solutions is $\mathbb{P}^1$ with 4 marked points $b_1^4 = -1$, where the additional states are possible according to the last row of (51). Thus, one can think of the $\mathbb{P}^1$ as a target space of the model where the center of mass of the string is allowed to move.

This interpretation is not quite correct however because we did not resolve the orbifold singularity completely. One can easy to see from (15), (20), (46) or (47) that the subgroup

$$\mathbb{Z}_2 \subset \mathbb{Z}_4, \tag{52}$$

is acting on the sections of the chiral de Rham complex over the each open set. But the action is nontrivial only along the fibers of the $O(2)$-bundle so the base $\mathbb{P}^1$ is the fixed point set of the action. Therefore we should consider the target space of the model as 2 copies of $\mathbb{P}^1$ (except probably the points $b_1^4 = -1$), where the second copy comes from the twisted sector. This picture is in agreement with the result of [11] where the chiral de Rham complex on the orbifolds has been introduced. It was shown there that twisted sectors chiral de Rham complex are the sheaves supported on the fixed points of the orbifold group action.

Thus, the natural suggestion is that we reproduce the geometry of 2-torus which double covers the $\mathbb{P}^1$ with ramification along the marked points $b_1^4 = -1$. It is evidently confirmed by the Hodge numbers calculation from (18): $h^{0,0} = h^{1,0} = h^{0,1} = h^{1,1} = 1$. Hence, adding the fermionic screening charge (44)we blow up the orbifold singularity of the Gepner model and obtain the σ-model on 2-torus which double covers $\mathbb{P}^1$.

4.2. $K > 2$ generalization.

In general case one has to deform Q^+ differential (35) adding the screening charge

$$Q^+ \to Q^+ + D_{orb},$$
$$D_{orb} = \oint dz \frac{1}{K}(\psi_1^* + ... + \psi_K^*) \exp(\frac{1}{K}(X_1^* + ... + X_K^*))(z) \tag{53}$$

which comes from the spectral flow operator $\prod_{i=1}^K U_i$. Similar to the $K = 2$ case there are also another fermionic screening charges commuting with $N = 2$ Virasoro superalgebra currents as well as with the charges Q_i^- but they do not appear as marginal operators of the model (18).

The set of screening charges $\{Q_1^+, ..., Q_K^+, D_{orb}\}$ defines the standard fan of the $O(K)$-bundle total space over $\mathbb{P}^{K-1}$. The highest dimensional cones σ_i of the fan are labeled by the differentials

$$D_i = Q_1^+ + ... + Q_{i-1}^+ + D_{orb} + Q_{i+1}^+, ..., Q_K^+, \; i = 1, ..., K \tag{54}$$

where Q_i^+ is missing. In the standard basis $(e_1, ..., e_K)$ of $\mathbb{R}^K$ the cones are generated by the set of vectors Σ_i

$$\Sigma_i = (s_1 = e_1, ..., s_{i-1} = e_{i-1}, s_i = \frac{1}{K}(e_1 + ... + e_K), s_{i+1} = e_{i+1}, ..., s_K = e_K) \tag{55}$$

Making the first substep of the cohomology calculation we obtain a $bc\beta\gamma$-system of fields associated to each differential D_i and the space of states generated by this system is the set of sections of chiral de Rham complex over the open set associated to the cone σ_i of the standard covering of the $O(K)$-bundle total space over $\mathbb{P}^{K-1}$. The analog of the formulas (46) can be written easily in terms of the dual basis $\check{\Sigma}_i$ to the Σ_i

$$\check{\Sigma}_i = (w_{(i)1}, ...w_{(i)K}), \; < w_{(i)j}, s_m >= \delta_{jm}, \tag{56}$$

Then the cohomology of D_i is generated by

$$\begin{aligned} b_{(i)j}(z) = \exp[w_{(i)j} \cdot X](z),\ \beta_{(i)j}(z) = w_{(i)j} \cdot \psi \exp[w_{(i)j} \cdot X](z), \\ b^*_{(i)j}(z) = (s_j \cdot \partial X^* - w_{(i)j} \cdot \psi s_j \cdot \psi^*) \exp[-w_{(i)j} \cdot X](z), \\ \beta^*_{(i)j}(z) = s_j \cdot \psi^* \exp[-w_{(i)j} \cdot X](z), \end{aligned} \tag{57}$$

where

$$\begin{aligned} b_{(i)i}(z) \leftrightarrow \ \textit{coordinate along the fiber},\ b_{(i)i} \\ b_{(i)j}(z),\ j \neq i \leftrightarrow \ \textit{coordinates along the base}\ b_{(i)j} \end{aligned} \tag{58}$$

The global sections of the chiral de Rham complex on the $O(K)$-bundle total space are given by Chech complex associated to the standard covering [3]. It finishes the first step of cohomology calculation.

In terms of the fields (57) the LG potential determined by the differential Q^- takes the form

$$W = (b_{(i)i})^2 (1 + \sum_{j \neq i} (b_{(i)j})^{2K}) \tag{59}$$

The $dW = 0$ points (Q^--cohomology) are given by the equations

$$\begin{aligned} b_{(i)i} = 0,\ \textit{when}\ \sum_{j \neq i} (b_{(i)j})^{2K} \neq -1, \\ (b_{(i)i})^2 = 0,\ \textit{when}\ \sum_{j \neq i} (b_{(i)j})^{2K} = -1, \end{aligned} \tag{60}$$

Thus the set of solutions is $\mathbb{P}^{K-1}$ with marked submanifold

$$\sum_{j \neq i} (b_{(i)j})^{2K} = -1, \tag{61}$$

where the additional states are possible according to the last row of (60).

Similar to the case $K = 2$ one can see that only the fields of the fiber are charged with respect to the operator $J[0]$ and the subgroup $\mathbb{Z}_2 \subset \mathbb{Z}_{2K}$ is acting non-trivialy along the fibers. Thus, the base $\mathbb{P}^{K-1}$ (considering as a zero section of the $O(K)$-bundle) is the fixed point set of the $\mathbb{Z}_2$-action and we conclude that the target space of the model is 2 copies of $\mathbb{P}^{K-1}$ (except the submanifold (61)), where the second copy comes from the twisted sector (see [11]).

Hence, it is natural to suggest that the geometry of the model is the $K-1$-dimensional CY manifold geometry which double covers the $\mathbb{P}^{K-1}$ with ramification along the submanifold (61). It is evidently confirmed by the Hodge numbers calculation from (18). For example, when $K = 3$

$$h^{0,0} = h^{2,0} = h^{0,2} = h^{2,2} = 1,\ h^{1,1} = 20 \tag{62}$$

which are Hodge numbers of $K3$. When $K = 4$ we find

$$\begin{aligned} h^{0,0} = h^{3,0} = h^{0,3} = h^{3,3} = h^{1,1} = h^{2,2} = 1, \\ h^{1,2} = h^{2,1} = 149 \end{aligned} \tag{63}$$

Thus, adding the fermionic screening charge (53) we blow up the orbifold singularity of the Gepner model and obtain the σ-model on the CY manifold which double covers $\mathbb{P}^{K-1}$.

It is important to note that in our free-field realization the center of mass of the string is allowed to move on the $\mathbb{P}^{K-1}$ which can be considered as the target space and hence we can interprate the model as a σ-model on $\mathbb{P}^{K-1}$. Though, the target space is not a CY manifold, nevertheless we have $N=2$ superconformal invariance. The possible solution of this puzzle is to consider these models as the examples of flux compactification [14], [15]. Moreover, the models considered here are very close to the known examples of the weak coupling limit of F-theory compactifications [16], [17]. The only difference is that they do not have the orientifolds planes. It is interesting to know if these models can be related with F-theory compactifications.

Finishing the Section we mention the question what is geometry of mirror models. It can be investigated by free-field $bc\beta\gamma$-representation but we left it for the future.

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